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14. ABSTRACT

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15. SUBJECT TERMS

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Remarks on the Polarimetric Radiative Transfer of the Atmosphere involving Ground Reflections

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ABSTRACT

The radiative transfer (RT) approach is widely used for studying scattering from layered random media with rough interfaces. Although it has been successful in several applications such as remote sensing of atmosphere it is well known that this approach involves certain approximations. In this paper these assumptions and approximations are reexamined and explained. To enable this a statistical approach is employed to this problem and the governing equations for the first and second moments of the wave fields are derived. A transition is hence made to arrive at a system of equations corresponding to that of the RT approach. It is thus found that more conditions are implicitly involved in the RT approach than generally believed to be necessary.

Keywords: random media, rough surfaces, radiative transfer, multilayer, wave approach

1. INTRODUCTION

Atmosphere is often modelled as a stratified medium whose scattering and absorption characteristics are continuous function of height. The refractive index of atmosphere hence has a continuously varying deterministic part with a relatively small randomly fluctuating part. In addition we may have to take into consideration reflection from the irregular ground. In this connection we are interested in studying scattering from a layered random medium with rough interfaces. This is a problem that one encounters often in many applications besides atmospheric sensing. A simple approach is to incoherently add the contributions of volumetric and surface fluctuations.[1–3] However, this is valid only when we are in the single scattering regime. There are some other hybrid approaches[4] which take into consideration some multiple scattering effects. Brown[5] outlines an iterative procedure which properly includes all multiple scattering interactions. However, it does not appear feasible to carry out the calculation beyond one or two iterations. Among the other methods currently used, perhaps the most widely used approach is the radiative transfer (RT) approach.[6–9] Here one formulates the scattering and propagation in each layer by using the radiative transfer equation which involves only the parameters of the medium of that layer. The boundary conditions are derived separately using some asymptotic procedure developed in rough surface scattering theory.[10–12] The RT equations, along with the boundary conditions, comprise the system that describes the problem.

Although this procedure appears to be reasonable and sound it is apparent that certain approximations are involved and we would like to know the conditions under which this kind of approach is appropriate. One way to better understand the RT approach is to compare it with the more rigorous wave approach. For the case of unbounded random media it was found that the RT approach is applicable when (a) one uses quasi-uniform field approximation, (b) the medium is statistically quasi-homogeneous, and (c) one uses the ladder approximation to the intensity operator of the Bethe-Salpeter equation.[13]

However, our problem has bounded structures which are randomly rough. Therefore it remains to be seen whether the conditions arrived at in the case of unbounded random media will be sufficient for our problem.

In this paper we employ a wave approach using surface scattering operators[14] to derive the transport equations for our multilayer problem. In this process we find that there are more conditions implied when we choose to apply the RT approach to our problem than it is widely believed to be necessary. One such condition is the weak surface correlation approximation. This means that the RT approach places certain restrictions on the type of rough interfaces that it can model accurately.

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The paper is organized as follows. In Section II we describe the geometry of the problem. In the next section we describe the RT approach to the problem. Next we describe the wave approach to this problem. In Section V we transition to the transport equation system. The paper concludes with a discussion of our findings.

2. DESCRIPTION OF THE PROBLEM

The geometry of the problem is shown in Figure 1. We have an N-layer random medium stack with rough interfaces. The permittivity of the j-th layer is $\epsilon_j + \tilde{\epsilon}_j(\mathbf{r})$ where ϵ_j is the deterministic part and $\tilde{\epsilon}_j$ is the randomly fluctuating part. The permeability of all the layers is that of free space. The randomly rough interfaces are given as $z = z_j + \zeta_j(\mathbf{r}_\perp)$. It is assumed that $\tilde{\epsilon}_j$ and ζ_j are zero-mean isotropic stationary random processes independent of each other. Thus, on the average the interfaces are parallel planes. Let $z_0 = 0$, and let d_j be the thickness of the j-th layer. The media above and below the stack are homogeneous with parameters ϵ_0, k_0, and ϵ_{N+1}, k_{N+1}, respectively. This system is excited by a monochromatic electromagnetic plane wave and we are interested in formulating the resulting multiple scattering process.

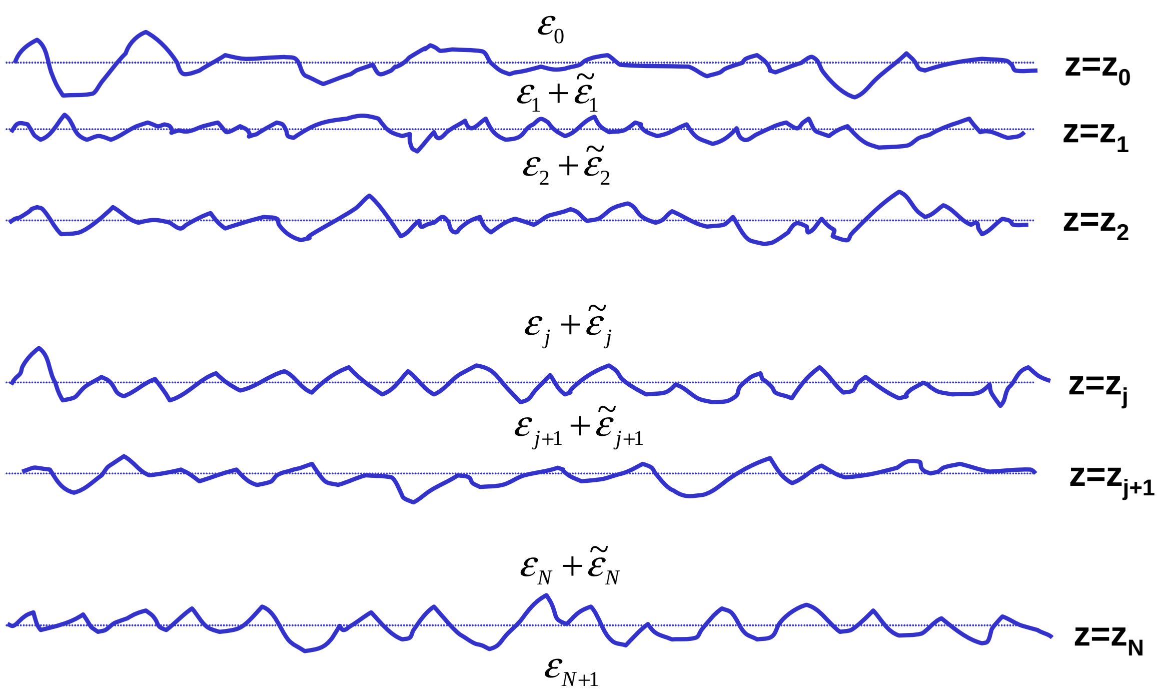

Figure 1. Geometry of the problem.

3. RADIATIVE TRANSFER APPROACH

Multiple scattering in a complex environment is well described by the radiative transfer theory. This theory is not only conceptually simple but also very efficient. The fundamental quantity here is the specific intensity $\mathbf{I}$ which is governed by the following equation[15–17]

$$\hat{s} \cdot \nabla \mathbf{I}(\mathbf{r}, \hat{s}) + \overline{\overline{\gamma}}\, \mathbf{I}(\mathbf{r}, \hat{s}) = \int d\Omega' \, \overline{\overline{\mathbf{P}}}\, (\hat{s}, \hat{s}')\mathbf{I}(\mathbf{r}, \hat{s}'). \tag{1}$$

One may regard this equation as a statement of conservation of energy density $\mathbf{I}$ which is a phase-space quantity at position $\mathbf{r}$ and direction $\hat{s}$. $\overline{\overline{\gamma}}$ is the extinction matrix which is a measure of loss of energy due to scattering in other directions. $\overline{\overline{\mathbf{P}}}$ is the phase matrix representing increase in energy density due to scattering from neighbouring

elements. Ω is the solid angle subtended by $\hat{s}$. Given the statistical characteristics of the medium one can readily calculate the phase function. The extinction matrix is hence calculated using the optical theorem. The specific intensity in each layer is governed by an equation similar to (1). Since our layer problem has translational invariance in azimuth the RT equation for the m-th layer takes the following form,

$$\cos\theta \frac{d}{dz}\mathbf{I}_m(z,\hat{s}) + \bar{\bar{\gamma}}_m\, \mathbf{I}_m(z,\hat{s}) = \int_{\Omega_m} d\Omega'\, \bar{\bar{\mathbf{P}}}_m\,(\hat{s},\hat{s}')\mathbf{I}_m(z,\hat{s}') \tag{2}$$

where the subscript m denotes that the quantity corresponds to those of the m-th layer and θ is the elevation angle of $\hat{s}$. This set of RT equations is complemented by a set of boundary conditions which are in turn based on energy conservation considerations. To be more precise we impose the condition that the energy flux density at each interface is conserved. This leads to the following boundary conditions on the m-th interface

$$\mathbf{I}_m^u(z_m,\hat{s}) = \int d\Omega'\, \bar{\bar{\mathbf{R}}}_{m+1,m}\,(\hat{s},\hat{s}')\mathbf{I}_m^d(z_m,\hat{s}') + \int d\Omega'\, \bar{\bar{\mathbf{T}}}_{m,m+1}\,(\hat{s},\hat{s}')\mathbf{I}_{m+1}^u(z_m,\hat{s}'). \tag{3}$$

The boundary conditions on the $(m-1)$-th interface are given as

$$\mathbf{I}_m^d(z_{m-1},\hat{s}) = \int d\Omega'\, \bar{\bar{\mathbf{R}}}_{m-1,m}\,(\hat{s},\hat{s}')\mathbf{I}_m^u(z_{m-1},\hat{s}') + \int d\Omega'\, \bar{\bar{\mathbf{T}}}_{m,m-1}\,(\hat{s},\hat{s}')\mathbf{I}_{m-1}^d(z_{m-1},\hat{s}'), \tag{4}$$

where $\bar{\bar{\mathbf{R}}}_{mn}$ and $\bar{\bar{\mathbf{T}}}_{mn}$ are the local reflection and transmission Mueller matrices. To be more specific, $\bar{\bar{\mathbf{R}}}_{mn}$ represents the reflection matrix of waves incident from medium n on the interface separating medium m and medium n. The superscripts u and d indicate whether the intensity corresponds to a wave travelling upwards or downwards. The integration in these expressions are over a solid angle (hemisphere) corresponding to $\hat{s}'$. Suppose we have a plane wave incident on this stack from above. Then the downward travelling intensity in Region 0 is

$$\mathbf{I}_0^d(z,\hat{s}) = \mathbf{B}_0\delta\left(\cos\theta_0 - \cos\theta_i\right)\delta(\phi_0 - \phi_i), \tag{5}$$

where $\mathbf{B}_0$ is the intensity of the incident plane wave and $\{\theta_i, \phi_i\}$ describes its direction. Since there is no source or scatterer in Region $N+1$,

$$\mathbf{I}_{N+1}^u(z,\hat{s}) = 0.$$

Notice again that these boundary conditions represent conservation of intensity at the interfaces. We should point out that the radiative transfer approach as applied to a particular problem is only a model based on certain assumptions. Since the RT theory is used in a variety of applications, the particular assumptions involved are described in terms of different terminologies, specific to the discipline where it is used. One good way to understand in more general terms the RT approach and the underlying assumptions is to compare it with the more rigorous wave approach. For the case of an unbounded random medium this kind of study was carried out in the seventies.[13] From that study we learn that the radiative transfer theory can be applied under the following conditions:

1. Statistical homogeneity of the medium fluctuations.
2. Quasi-stationary field approximation.
3. Weak fluctuations:
   (a) Ladder approximation to the intensity operator.
   (b) Kraichnan approximation to the mass operator.

These are the well-known conditions that we associate with the RT approach. However, our problem has bounded structures and, further, they are randomly rough. The question is this: are the above conditions sufficient to apply the RT approach for our problem? This is the motivation for this paper. We follow the wave approach to this problem, derive the equations for the intensities, and hence make the transition to the RT equations. This procedure enables us to better understand the necessary conditions for using the RT approach for our problem.

4. WAVE APPROACH

The following are the equations that govern the waves in the layer structure:

$$\nabla \times \nabla \times \mathbf{E}_j - k_j^2\mathbf{E}_j = v_j\mathbf{E}_j \qquad j = 1, \cdots, N, \tag{6}$$

where $v_j \equiv \omega^2 \mu \tilde{\epsilon}_j(\mathbf{r})$ represents the volumetric fluctuation in Region j. For the homogeneous regions above and below we have

$$\nabla \times \nabla \times \mathbf{E}_0 - k_0^2 \mathbf{E}_0 = 0, \tag{7a}$$

$$\nabla \times \nabla \times \mathbf{E}_{N+1} - k_{N+1}^2 \mathbf{E}_{N+1} = 0. \tag{7b}$$

The boundary conditions at the j-th interface are

$$\hat{n} \times \mathbf{E}_j(\mathbf{r}_\perp, \zeta_j) = \hat{n} \times \mathbf{E}_{j+1}(\mathbf{r}_\perp, \zeta_j), \tag{8a}$$

$$\hat{n} \times \nabla \times \mathbf{E}_j(\mathbf{r}_\perp, \zeta_j) = \hat{n} \times \nabla \times \mathbf{E}_{j+1}(\mathbf{r}_\perp, \zeta_j). \tag{8b}$$

where $\hat{n}$ is the unit vector normal to the j-th interface with normal pointing into the medium j. This system is complemented by the radiation conditions well away from the stack. We assume that we know the solution to the problem without volumetric fluctuations, and denote it as $\check{\mathbf{E}}$. Let the Green's functions to this problem be denoted as $\check{\mathbf{G}}_{ij}$. These Green's functions are governed by the following set of equations:

$$\nabla \times \nabla \times \check{\mathbf{G}}_{jk}(\mathbf{r}, \mathbf{r}') - k_j^2 \check{\mathbf{G}}_{jk}(\mathbf{r}, \mathbf{r}') = \bar{\bar{\mathbf{I}}} \delta_{jk} \delta(\mathbf{r} - \mathbf{r}'), \tag{9a}$$

$$\hat{n} \times \check{\mathbf{G}}_{jk}(\mathbf{r}_\perp, \zeta_j; \mathbf{r}') = \hat{n} \times \check{\mathbf{G}}_{(j+1)k}(\mathbf{r}_\perp, \zeta_j; \mathbf{r}'), \tag{9b}$$

$$\hat{n} \times \nabla \times \check{\mathbf{G}}_{jk}(\mathbf{r}_\perp, \zeta_j; \mathbf{r}') = \hat{n} \times \nabla \times \check{\mathbf{G}}_{(j+1)k}(\mathbf{r}_\perp, \zeta_j; \mathbf{r}'). \tag{9c}$$

Another pair of equations similar to (9b) and (9c) corresponding to $(j-1)$-th interface need to be added this list. $\bar{\bar{\mathbf{I}}}$ here is unit dyad. Using these Green's functions and the radiation conditions the wave functions can be represented as

$$\mathbf{E}_j(\mathbf{r}) = \check{\mathbf{E}}_j(\mathbf{r}) + \sum_{k=1}^{N} \int_{\Omega_k} d\mathbf{r}' \check{\mathbf{G}}_{jk}(\mathbf{r}, \mathbf{r}') v_k(\mathbf{r}') \mathbf{E}_k(\mathbf{r}') \qquad j = 0, 1, \cdots, N+1 \tag{10}$$

where $\Omega_k = \{\mathbf{r}'_\perp; \zeta_k < z' < \zeta_{k-1}\}$. Note that $v_0 = v_{N+1} = 0$. We first average (10) w.r.t. volumetric fluctuations to get

$$\langle \mathbf{E}_j(\mathbf{r}) \rangle_v = \check{\mathbf{E}}_j(\mathbf{r}) + \sum_{k=1}^{N} \sum_{l=1}^{N} \int_{\Omega_k} d\mathbf{r}' \int_{\Omega_l} d\mathbf{r}'' \check{\mathbf{G}}_{jk}(\mathbf{r}, \mathbf{r}') \langle \mathbf{G}_{kl}(\mathbf{r}', \mathbf{r}'') \rangle_v \langle v_k(\mathbf{r}') v_l(\mathbf{r}'') \rangle \langle \mathbf{E}_l(\mathbf{r}'') \rangle_v. \tag{11}$$

The subscript v is used to denote averaging with respect to volumetric fluctuations. Here we have used a first order approximation to the mass operator based on weak fluctuations. The volumetric fluctuations in different regions are assumed to be uncorrelated, which means that

$$\langle v_k(\mathbf{r}') v_l(\mathbf{r}'') \rangle = \delta_{kl} C_k(\mathbf{r}' - \mathbf{r}''), \tag{12}$$

where C_k is the correlation function of the volumetric fluctuations in Region k. Inserting (12) in (11) and employing $\nabla \times \nabla \times \bar{\bar{\mathbf{I}}} - k_j^2$ on (11) we get

$$\nabla \times \nabla \times \langle \mathbf{E}_j(\mathbf{r}) \rangle_v - k_j^2 \langle \mathbf{E}_j(\mathbf{r}) \rangle_v = \int_{\Omega_j} d\mathbf{r}' \langle \mathbf{G}_{jj}(\mathbf{r}, \mathbf{r}') \rangle_v C_j(\mathbf{r} - \mathbf{r}') \langle \mathbf{E}_j(\mathbf{r}') \rangle_v. \tag{13}$$

Next we average (13) over the surface fluctuations,

$$\nabla \times \nabla \times \langle \mathbf{E}_j(\mathbf{r}) \rangle_{vs} - k_j^2 \langle \mathbf{E}_j(\mathbf{r}) \rangle_{vs} = \int_{\bar{\Omega}_j} d\mathbf{r}' \Big\langle \langle \mathbf{G}_{jj}(\mathbf{r}, \mathbf{r}') \rangle_v C_j(\mathbf{r} - \mathbf{r}') \langle \mathbf{E}_j(\mathbf{r}') \rangle_v \Big\rangle_s$$

where the subscript s denotes averaging over surface fluctuations and $\bar{\Omega}_j = \{\mathbf{r}'_\perp; \hat{z}_j < z' < \hat{z}_{j-1}\}$. We approximate $\Big\langle \langle \mathbf{G}_{jj}(\mathbf{r}, \mathbf{r}') \rangle_v C_j(\mathbf{r} - \mathbf{r}') \langle \mathbf{E}_j(\mathbf{r}') \rangle_v \Big\rangle_s$ as $\langle \mathbf{G}_{jj}(\mathbf{r}, \mathbf{r}') \rangle_{vs} C_j(\mathbf{r} - \mathbf{r}') \langle \mathbf{E}_j(\mathbf{r}') \rangle_{vs}$ and obtain

$$\nabla \times \nabla \times \langle \mathbf{E}_j(\mathbf{r}) \rangle_{vs} - k_j^2 \langle \mathbf{E}_j(\mathbf{r}) \rangle_{vs} = \int_{\bar{\Omega}_j} d\mathbf{r}' \langle \mathbf{G}_{jj}(\mathbf{r}, \mathbf{r}') \rangle_{vs} C_j(\mathbf{r} - \mathbf{r}') \langle \mathbf{E}_j(\mathbf{r}') \rangle_{vs}. \tag{14}$$

We call this the weak surface correlation approximation. We will later find that this is one additional approximation necessary to arrive at the RT system. First note from (14) that $(\nabla\times\nabla\times \bar{\bar{\mathbf{I}}} - k_j^2)\langle \mathbf{E}_j(\mathbf{r})\rangle_{vs} = 0$ for $j = 0, N+1$. This means that the coherent propagation constants in regions above and below the layer stack are unaffected by the fluctuations of the problem. However, they indeed get modified within the stack region. On writing (14) as $(\nabla \times \nabla\times \bar{\bar{\mathbf{I}}} - k_j^2 - \mathcal{L})\langle\psi_j\rangle = 0$, where $\mathcal{L}$ denotes the integral operator $\int_{\bar{\Omega}_j} d\mathbf{r}' \, \langle \mathbf{G}_{jj}(\mathbf{r},\mathbf{r}')\rangle_{vs} C_j(\mathbf{r}-\mathbf{r}')$, we infer that $\chi_j \equiv \sqrt{(k_j^2 + \mathcal{L})}$ represents the mean propagation constant in $\bar{\Omega}_j$. Observe that χ_j depends explicitly on the volumetric fluctuations in Region j and implicitly on the fluctuations of the stack, both volumetric and surface. This is in contrast to the RT approach where $\bar{\bar{\gamma}}_j$ depends exclusively on the volumetric fluctuations in Region j. Moreover, χ_j depends on the polarization if the fluctuations of the problem are anisotropic. Further, even if the volumetric fluctuations are isotropic χ_j will be polarization dependent because of surface reflections. This is in contrast to the RT approach where $\bar{\bar{\gamma}}_j$ is polarization dependent only when the volumetric fluctuations are anisotropic.

Since the problem is statistically homogeneous in azimuth the mean fields in our system have the following form:

$$\langle \mathbf{E}_j^p(\mathbf{r})\rangle_{vs} = \exp(i\mathbf{k}_{\perp i}\cdot\mathbf{r})\Big\{A_j^p(\mathbf{k}_{\perp i})\mathbf{p}_j^+ \exp[iq_j^p z] + B_j^p(\mathbf{k}_{\perp i})\mathbf{p}_j^- \exp[-iq_j^p z]\Big\} \qquad j = 1, 2, \cdots, N, \tag{15}$$

$$\langle \mathbf{E}_0^p(\mathbf{r})\rangle_{vs} = \exp(i\mathbf{k}_{\perp i}\cdot\mathbf{r})\Big\{\mathbf{p}_0^- \exp[-ik_{0zi}z] + R^p(\mathbf{k}_{\perp i})\mathbf{p}_0^+ \exp[ik_{0zi}z]\Big\}, \tag{16}$$

and

$$\langle \mathbf{E}_{N+1}^p(\mathbf{r})\rangle_{vs} = \exp(i\mathbf{k}_{\perp i}\cdot\mathbf{r})T^p(\mathbf{k}_{\perp i})\mathbf{p}_{N+1}^- \exp[-ik_{(N+1)zi}z], \tag{17}$$

where the superscript p stands for the polarization, either horizontal or vertical. $\mathbf{p}$ is the unit vector representing polarization. q_j is the z-component of χ_j. The subscript i is used to indicate that the wave vector is in the incident direction. R and T denote respectively the mean reflection and transmission coefficient of the stack. A_j and B_j denote respectively the mean coefficients of up-going and down-going waves in the j-th layer. Based on this we can formulate the waves averaged w.r.t. volumetric fluctuations as

$$\langle \mathbf{E}_j(\mathbf{r})\rangle_v^p = \frac{1}{4\pi^2}\int d\mathbf{k}_\perp \exp(i\mathbf{k}_\perp\cdot\mathbf{r})\Big\{A_j^{pq}(\mathbf{k}_\perp,\mathbf{k}_{\perp i})\mathbf{q}_j^+ \exp[iq_j z] + B_j^{pq}(\mathbf{k}_\perp,\mathbf{k}_{\perp i})\mathbf{q}_j^- \exp[-iq_j z]\Big\} \qquad j = 1, 2, \cdots, N, \tag{18}$$

$$\langle \mathbf{E}_0(\mathbf{r})\rangle_v^p = \exp(i\mathbf{k}_{\perp i}\cdot\mathbf{r})\exp[-ik_{0zi}z]\mathbf{p}_0^- + \frac{1}{4\pi^2}\int d\mathbf{k}_\perp \exp(i\mathbf{k}_\perp\cdot\mathbf{r})R^{pq}(\mathbf{k}_\perp,\mathbf{k}_{\perp i})\mathbf{q}_0^+ \exp[ik_{0z}z], \tag{19}$$

and

$$\langle \mathbf{E}_{N+1}(\mathbf{r})\rangle_v^p = \frac{1}{4\pi^2}\int d\mathbf{k}_\perp \exp(i\mathbf{k}_\perp\cdot\mathbf{r})T^{pq}(\mathbf{k}_\perp,\mathbf{k}_{\perp i})\mathbf{q}_{N+1}^- \exp[-ik_{(N+1)z}z]. \tag{20}$$

where A_j, B_j, R and T are now integral operators representing scattering from rough interfaces. The boundary conditions associated with the above equations at the j-th interface are

$$\hat{n}\times\langle \mathbf{E}_j(\mathbf{r}_\perp,\zeta_j)\rangle_v = \hat{n}\times\langle \mathbf{E}_{j+1}(\mathbf{r}_\perp,\zeta_j)\rangle_v \qquad j = 1, 2, \cdots, N \tag{21a}$$

and

$$\hat{n}\times\nabla\times\langle \mathbf{E}_j(\mathbf{r}_\perp,\zeta_j)\rangle_v = \hat{n}\times\nabla\times\langle \mathbf{E}_{j+1}(\mathbf{r}_\perp,\zeta_j)\rangle_v \qquad j = 1, 2, \cdots, N. \tag{21b}$$

The above system may be solved either numerically or by anyone of the asymptotic methods available in rough surface scattering theory[10–12] to evaluate the mean coefficients that appear in (15)-(17).

We proceed now to the analysis of the second moments, by starting with (10). For convenience we write it in symbolic form as

$$\mathbf{E}_j = \check{\mathbf{E}}_j + \sum_{k=1}^{N}\check{\mathbf{G}}_{jk}v_k\mathbf{E}_k. \tag{22}$$

We take the tensor product of this equation with its complex conjugate and average w.r.t. volumetric fluctuations and obtain

$$\left\langle \mathbf{E}_j \otimes \mathbf{E}_j^* \right\rangle_v = \langle \mathbf{E}_j \rangle_v \otimes \langle \mathbf{E}_j^* \rangle_v + \sum_{k=1}^{N} \sum_{k'=1}^{N} \sum_{l=1}^{N} \sum_{l'=1}^{N} \langle \mathbf{G}_{jk} \rangle_v \otimes \langle \mathbf{G}_{jk'}^* \rangle_v \mathbf{K}_{kk'll'} \left\langle \mathbf{E}_l \otimes \mathbf{E}_{l'}^* \right\rangle_v , \tag{23}$$

where $\mathbf{K}$ is the intensity operator of the volumetric fluctuations. Employing the weak fluctuation approximation we approximate $\mathbf{K}$ by its leading term

$$\mathbf{K}_{kk'll'} \simeq \langle v_k \otimes v_k^* \rangle \delta_{kk'll'} \ \bar{\bar{\mathbf{I}}} \, . \tag{24}$$

Next we average (23) w.r.t. the surface fluctuations and employ the weak surface correlation approximation as before to get

$$\left\langle \mathbf{E}_j \otimes \mathbf{E}_j^* \right\rangle_{vs} = \left\langle \langle \mathbf{E}_j \rangle_v \otimes \langle \mathbf{E}_j^* \rangle_v \right\rangle_s + \sum_{k=1}^{N} \left\langle \, \langle \mathbf{G}_{jk} \rangle_v \otimes \left\langle \mathbf{G}_{jk}^* \right\rangle_v \, \right\rangle_s \langle v_k \otimes v_k^* \rangle \left\langle \mathbf{E}_k \otimes \mathbf{E}_k^* \right\rangle_{vs} . \tag{25}$$

The above is an equation for the second moment of the wave function $\mathbf{E}$, which can be decomposed into a coherent part $\bar{\mathbf{E}}$ and a diffuse part $\tilde{\mathbf{E}}$. Therefore,

$$\langle \mathbf{E} \otimes \mathbf{E}^* \rangle = \bar{\mathbf{E}} \otimes \bar{\mathbf{E}}^* + \langle \tilde{\mathbf{E}} \otimes \tilde{\mathbf{E}}^* \rangle. \tag{26}$$

The coherent part is not of much interest; we know that it is specular for our problem. The diffuse or the incoherent part is of more interest. Therefore we write (25) in terms of diffuse fields:

$$\left\langle \tilde{\mathbf{E}}_j \otimes \tilde{\mathbf{E}}_j^* \right\rangle = \left\langle \widetilde{\langle \mathbf{E}_j \rangle}_v \otimes \widetilde{\langle \mathbf{E}_j^* \rangle}_v \right\rangle_s + \sum_{k=1}^{N} \left\langle \, \langle \mathbf{G}_{jk} \rangle_v \otimes \left\langle \mathbf{G}_{jk}^* \right\rangle_v \, \right\rangle_s \langle v_k \otimes v_k^* \rangle \left\langle \mathbf{E}_k \otimes \mathbf{E}_k^* \right\rangle_{vs} , \tag{27}$$

where $\widetilde{\langle \mathbf{E}_j \rangle}_v$ is the fluctuating part of $\langle \mathbf{E}_j \rangle_v$. Let us now write (27) in more detail as:

$$\left\langle \tilde{\mathbf{E}}_j(\mathbf{r}) \otimes \tilde{\mathbf{E}}_j^*(\mathbf{r}') \right\rangle = \left\langle \widetilde{\langle \mathbf{E}_j(\mathbf{r}) \rangle}_v \otimes \widetilde{\langle \mathbf{E}_j^*(\mathbf{r}') \rangle}_v \right\rangle_s +$$

$$+ \sum_{k=1}^{N} \int_{\Omega_k} d\mathbf{r}_1 \int_{\Omega_k} d\mathbf{r}_1' \left\langle \, \langle \mathbf{G}_{jk}(\mathbf{r}, \mathbf{r}_1) \rangle_v \otimes \left\langle \mathbf{G}_{jk}^*(\mathbf{r}', \mathbf{r}_1') \right\rangle_v \, \right\rangle_s |k_k|^4 C_k(\mathbf{r}_1 - \mathbf{r}_1') \rangle \left\langle \mathbf{E}_k(\mathbf{r}_1) \otimes \mathbf{E}_k^*(\mathbf{r}_1') \right\rangle_{vs} . \tag{28}$$

As it stands this equation is very difficult to solve either analytically or numerically. Besides, one important goal for us is to investigate the conditions needed for employing the radiative transfer approach. With this in mind we introduce Wigner transforms. Note that (28) is an equation for the coherence function which is a 'space-space' quantity. On the other hand the RT equation, as we saw earlier, is an equation for the specific intensity which is a 'phase-space' quantity. Wigner transforms serve as a bridge to link these two quantities.[18–21]

We introduce Wigner transforms of waves and Green's functions as

$$\mathcal{E}_m \left(\frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{k} \right) = \int d(\mathbf{r} - \mathbf{r}') \left\langle \mathbf{E}_m(\mathbf{r}) \otimes \mathbf{E}_m^*(\mathbf{r}') \right\rangle e^{-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}, \tag{29}$$

$$\mathcal{G}_{mn} \left(\frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{k} \middle| \frac{\mathbf{r}_1 + \mathbf{r}_1'}{2}, \mathbf{l} \right) = \int d(\mathbf{r} - \mathbf{r}') \int d(\mathbf{r}_1 - \mathbf{r}_1') e^{-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} e^{i\mathbf{l} \cdot (\mathbf{r}_1 - \mathbf{r}_1')} \left\langle \, \langle \mathbf{G}_{mn}(\mathbf{r}, \mathbf{r}_1) \rangle_v \otimes \langle \mathbf{G}_{mn}^*(\mathbf{r}', \mathbf{r}_1') \rangle_v \, \right\rangle_s . \tag{30}$$

In terms of these transforms (28) becomes

$$\tilde{\mathcal{E}}_m(\mathbf{r}, \mathbf{k}) = \tilde{\mathcal{E}}_m^s(\mathbf{r}, \mathbf{k}) + \frac{1}{(2\pi)^6} \sum_{n=1}^{N} |k_n|^4 \int_{\Omega_n} d\mathbf{r}' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta} \ \mathcal{G}_{mn}(\mathbf{r}, \mathbf{k} | \mathbf{r}', \boldsymbol{\alpha}) \Phi_n(\boldsymbol{\alpha} - \boldsymbol{\beta}) \mathcal{E}_n(\mathbf{r}', \boldsymbol{\beta}), \tag{31}$$

where Φ_n is the spectral density of the volumetric fluctuations in the n-th layer. We have used the superscript s in the first term to indicate that this is due to surface scattering as defined by the first term in (28).

The fact that our problem has translational invariance in azimuth implies the following:

$$\mathcal{E}_m(\mathbf{r}, \mathbf{k}) = \mathcal{E}_m(z, \mathbf{k}), \tag{32a}$$

$$\mathcal{G}_{mn}(\mathbf{r}, \mathbf{k}|\mathbf{r}', \mathbf{l}) = \mathcal{G}_{mn}(z, \mathbf{k}|z', \mathbf{l}; \mathbf{r}_\perp - \mathbf{r}'_\perp). \tag{32b}$$

Using these relations in (31) we have

$$\tilde{\mathcal{E}}_m(z, \mathbf{k}) = \tilde{\mathcal{E}}^s_m(z, \mathbf{k}) + \frac{1}{(2\pi)^6}\sum_{n=1}^{N} |k_n|^4 \int_{z_n}^{z_{n-1}} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\; \mathcal{G}_{mn}(z, \mathbf{k}|z', \boldsymbol{\alpha}; 0)\Phi_n(\boldsymbol{\alpha} - \boldsymbol{\beta})\mathcal{E}_n(z', \boldsymbol{\beta}), \tag{33}$$

where $\mathcal{G}_{mn}(z, \mathbf{k}|z', \boldsymbol{\alpha}; 0)$ is the Fourier transform of $\mathcal{G}_{mn}(z, \mathbf{k}|z', \boldsymbol{\alpha}; \mathbf{r}_\perp - \mathbf{r}'_\perp)$ w.r.t. $\mathbf{r}_\perp - \mathbf{r}'_\perp$ evaluated at the origin of the spectral space. To proceed further we need to evaluate $\mathcal{G}_{mn}$. Further we need to relate this system with that of RT, which involves the boundary conditions at the interfaces. In view of this we need to identify the coherence functions corresponding to up- and down-going wave functions. To facilitate this we decompose $\langle G_{mn}\rangle_v$ into its components,

$$\langle \mathbf{G}_{mn}\rangle_v = \delta_{mn}\mathbf{G}^o_m + \mathbf{G}^{uu}_{mn} + \mathbf{G}^{ud}_{mn} + \mathbf{G}^{du}_{mn} + \mathbf{G}^{dd}_{mn}, \tag{34}$$

where the first term is the singular part of the Green's function. The superscripts u and d indicate up- and down-going elements of the waves. The other components are due to reflections from boundaries. These are formally constructed using the concept of surface scattering operators as follows,[12]

$$\left\langle G^{ab}_{mn}(\mathbf{r}, \mathbf{r}')\right\rangle^{\mu\nu}_v = \frac{1}{(2\pi)^4}\int d\mathbf{k}_\perp \int d\mathbf{k}'_\perp \left\{S^{ab}_{mn}(\mathbf{k}_\perp, \mathbf{k}'_\perp)\right\}^{\mu\nu} e^{i\mathbf{k}_\perp\cdot\mathbf{r} + iaq^\mu_m(\mathbf{k}_\perp)z} e^{-i\mathbf{k}'_\perp\cdot\mathbf{r}' - ibq^\nu_n(\mathbf{k}'_\perp)z'}, \tag{35}$$

where $\mathbf{S}^{ab}_{mn}$ is the surface scattering operator. The superscripts a and b on S are used to indicate whether the waves are up-going or down-going. In the exponents, $a, b = 1$ if the waves are up-going. We let $a, b = -1$ if the waves are down-going. The z-component of the mean propagation constants in the n-th layer is denoted as q_n. We recall that $\mathcal{G}_{mn}$ is the Wigner transform of $\left\langle \langle \mathbf{G}_{mn}\rangle_v \otimes \langle \mathbf{G}^*_{mn}\rangle_v \right\rangle_s$. The superscripts μ, ν stand for polarization, either h or v. When we use (34) to perform the Wigner transform we ignore all cross terms. In other words, we make the following approximation,

$$\mathcal{G}_{mn} \simeq \delta_{mn}\mathcal{G}^o_m + \mathcal{G}^{uu}_{mn} + \mathcal{G}^{ud}_{mn} + \mathcal{G}^{du}_{mn} + \mathcal{G}^{dd}_{mn},$$

where $\mathcal{G}^{ab}_{mn}$ is the Wigner transform of $\left\langle \left\langle \mathbf{G}^{ab}_{mn}\right\rangle_v \otimes \left\langle \mathbf{G}^{ab*}_{mn}\right\rangle_v \right\rangle_s$. This approximation reminds us of the concept of incoherent addition of intensities associated with the radiative transfer theory.

With the introduction of this representation for $\mathcal{G}_{mn}$ in (31) we can trace up- and down-going waves to obtain the following equations for the coherence function:

$$\begin{aligned}\tilde{\mathcal{E}}^u_m(z, \mathbf{k}) = \tilde{\mathcal{E}}^{su}_m(z, \mathbf{k}) &+ \frac{1}{(2\pi)^6}|k_m|^4 \int_{z_m}^{z} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\;\; \mathcal{G}^{>}_m(z, \mathbf{k}|z', \boldsymbol{\alpha}; 0)\Phi_m(\boldsymbol{\alpha} - \boldsymbol{\beta})\mathcal{E}_m(z', \boldsymbol{\beta}) \\ &+ \frac{1}{(2\pi)^6}\sum_{n=1}^{N} |k_n|^4 \int_{z_n}^{z_{n-1}} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\;\; \mathcal{G}^{ua}_{mn}(z, \mathbf{k}|z', \boldsymbol{\alpha}; 0)\Phi_n(\boldsymbol{\alpha} - \boldsymbol{\beta})\mathcal{E}^a_n(z', \boldsymbol{\beta})\end{aligned} \tag{36a}$$

$$\begin{aligned}\tilde{\mathcal{E}}^d_m(z, \mathbf{k}) = \tilde{\mathcal{E}}^{sd}_m(z, \mathbf{k}) &+ \frac{1}{(2\pi)^6}|k_m|^4 \int_{z}^{z_{m-1}} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\; \mathcal{G}^{<}_m(z, \mathbf{k}|z', \boldsymbol{\alpha}; 0)\Phi_m(\boldsymbol{\alpha} - \boldsymbol{\beta})\mathcal{E}_m(z', \boldsymbol{\beta}) \\ &+ \frac{1}{(2\pi)^6}\sum_{n=1}^{N} |k_n|^4 \int_{z_n}^{z_{n-1}} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\; \mathcal{G}^{da}_{mn}(z, \mathbf{k}|z', \boldsymbol{\alpha}; 0)\Phi_n(\boldsymbol{\alpha} - \boldsymbol{\beta})\mathcal{E}^a_n(z', \boldsymbol{\beta})\end{aligned} \tag{36b}$$

Note that summation over $a = \{u, d\}$ is implied in the above equations. The first term in these equations, $\tilde{\mathcal{E}}^{sa}$, represents the contribution due exclusively to surface scattering, and has the following form:

$$\left\{\tilde{\mathcal{E}}^{sa}_m(z,\mathbf{k})\right\}^{\mu\nu} = 2\pi\delta\left\{k_z - \frac{1}{2}a\left[q^{\mu}_m(\mathbf{k}_\perp) + q^{\nu*}_m(\mathbf{k}_\perp)\right]\right\} e^{ia[q^{\mu}_m - q^{\nu*}_m]z} \left\langle \left\{\tilde{\Sigma}^a_m\right\}^{\mu\mu'} \left\{\tilde{\Sigma}^{a*}_m\right\}^{\nu\nu'} (\mathbf{k}_\perp, \mathbf{k}_{\perp i})\right\rangle_s E_{\mu' i} E^*_{\nu' i}, \tag{37}$$

where Σ^a_m is the amplitude of the up-going wave in the m-th layer after volumetric averaging is performed. This means that it is a random function of surface fluctuations. When we substitute (37) and the expressions for $\mathcal{G}_{mn}$ in (36) we find that

$$\left\{\tilde{\mathcal{E}}^a_m(z,\mathbf{k})\right\}_{\mu\nu} = 2\pi\delta\left\{k_z - \frac{1}{2}a[q^{\mu}_m(\mathbf{k}_\perp) + q^{\nu*}_m(\mathbf{k}_\perp)]\right\} e^{ia[q^{\mu}_m - q^{\nu*}_m]z} \left\{\tilde{\mathcal{E}}^a_m(z,\mathbf{k}_\perp)\right\}_{\mu\nu}. \tag{38}$$

On substituting this in (36) and differentiating w.r.t. z we obtain the following transport equations:

$$\begin{aligned}&\left\{\frac{d}{dz} + i\left[q_\mu(\mathbf{k}_\perp) - q^*_\nu(\mathbf{k}_\perp)\right]\right\} \tilde{\mathcal{E}}^u_{\mu\nu}(z,\mathbf{k}_\perp) = \tilde{\tilde{\mathcal{E}}}^u_{\mu\nu}(z,\mathbf{k}_\perp) + \\ &\quad + \frac{|k_m|^4}{(2\pi)^2}\int d\boldsymbol{\alpha}_\perp S^>_\mu \otimes S^{>*}_\nu \Phi_m \left\{\mathbf{k}_\perp - \boldsymbol{\alpha}_\perp; \frac{1}{2}\left[q_\mu(\mathbf{k}_\perp) + q^*_\nu(\mathbf{k}_\perp)\right] - \frac{1}{2}a\left[q_{\mu'}(\boldsymbol{\alpha}_\perp) + q^*_{\nu'}(\boldsymbol{\alpha}_\perp)\right]\right\} \tilde{\mathcal{E}}^a_{\mu'\nu'}(z,\boldsymbol{\alpha}_\perp),\end{aligned} \tag{39a}$$

$$\begin{aligned}&\left\{-\frac{d}{dz} + i\left[q_\mu(\mathbf{k}_\perp) - q^*_\nu(\mathbf{k}_\perp)\right]\right\} \tilde{\mathcal{E}}^d_{\mu\nu}(z,\mathbf{k}_\perp) = \tilde{\tilde{\mathcal{E}}}^d_{\mu\nu}(z,\mathbf{k}_\perp) + \\ &\quad + \frac{|k_m|^4}{(2\pi)^2}\int d\boldsymbol{\alpha}_\perp S^<_\mu \otimes S^{<*}_\nu \Phi_m \left\{\mathbf{k}_\perp - \boldsymbol{\alpha}_\perp; -\frac{1}{2}\left[q_\mu(\mathbf{k}_\perp) + q^*_\nu(\mathbf{k}_\perp)\right] - \frac{1}{2}a\left[q_{\mu'}(\boldsymbol{\alpha}_\perp) + q^*_{\nu'}(\boldsymbol{\alpha}_\perp)\right]\right\} \tilde{\mathcal{E}}^a_{\mu'\nu'}(z,\boldsymbol{\alpha}_\perp),\end{aligned} \tag{39b}$$

where $\tilde{\tilde{\mathcal{E}}}^a_{\mu\nu}$ represents scattering due to the coherent part of $\mathcal{E}$, whereas the integral term in (39) represents scattering due to the diffuse part of $\mathcal{E}$. We may also regard $\tilde{\tilde{\mathcal{E}}}^a_{\mu\nu}$ as the source to our transport equations; it is given as

$$\begin{aligned}\tilde{\tilde{\mathcal{E}}}^a_{\mu\nu} = |k_m|^4\, \Phi_m &\left\{\mathbf{k}_\perp - \mathbf{k}_{\perp i}; \frac{1}{2}a\left[q_\mu(\mathbf{k}_\perp) + q^*_\nu(\mathbf{k}_\perp)\right] - \frac{1}{2}b\left[q_\mu(\mathbf{k}_{\perp i}) + q^*_\nu(\mathbf{k}_{\perp i})\right]\right\} \\ &\left(\mathbf{S}^a_\mu \otimes \mathbf{S}^{a*}_\mu\right) : \left[\left(\langle \mathbf{S}^b_{\mu i}\rangle \cdot \mathbf{E}_i\rangle\right) \otimes \left(\langle \mathbf{S}^b_{\nu i}\rangle \cdot \mathbf{E}_i\rangle\right)^*\right].\end{aligned} \tag{40}$$

where summation over b is implied. Note that $\tilde{\mathcal{E}}^a$ in (39) includes both $\tilde{\mathcal{E}}^u$ and $\tilde{\mathcal{E}}^d$ (corresponding to up and down-going waves). When the superscripts $\{a, b\}$ corresponds to u the value of $\{a, b\}$ in the argument of Φ_m take the value $+1$; on the other hand when the superscripts $\{a, b\}$ corresponds to d the value of a in the argument of Φ_m take the value -1. Since all quantities in (39) and (40) correspond to the same layer m we have dropped the subscript m to avoid cumbersome notations. To obtain appropriate boundary conditions we have to go back to the integral equation representations for $\tilde{\mathcal{E}}^u_{\mu\nu}$ and $\tilde{\mathcal{E}}^d_{\mu\nu}$ and observe their behaviour at the interfaces and try to find a relation between them. After considerable manipulations we managed to arrive at the following boundary conditions. At the $(m-1)$-th interface we have

$$\tilde{\mathcal{E}}^d_m(z_{m-1},\mathbf{k}_\perp) = \int d\mathbf{k}'_\perp \left\langle \dddot{\mathcal{R}}_{m-1,m}\,(\mathbf{k}_\perp,\mathbf{k}'_\perp)\right\rangle \tilde{\mathcal{E}}^u_m(z_{m-1},\mathbf{k}'_\perp), \tag{41a}$$

with $\dddot{\mathcal{R}} = \ddot{\mathbf{R}} \otimes \dddot{\mathbf{R}}^*$ where $\ddot{\mathbf{R}}_{m-1,m}$ is the stack reflection matrix (not the local reflection matrix) for a wave incident from below on the $(m-1)$-th interface. Similarly

$$\tilde{\mathcal{E}}^u_m(z_{m-1},\mathbf{k}_\perp) = \int d\mathbf{k}'_\perp \left\langle \dddot{\mathcal{R}}_{m+1,m}\,(\mathbf{k}_\perp,\mathbf{k}'_\perp)\right\rangle \tilde{\mathcal{E}}^d_m(z_{m-1},\mathbf{k}'_\perp), \tag{41b}$$

where $\dddot{\mathcal{R}}_{m+1,m}$ is the tensor product of stack reflection matrix for a wave incident from above on the m-th interface. We were able to obtain the boundary conditions only after imposing certain approximations such as the one given below. Consider the following identity:

$$\mathbf{S}^{du}_{mm} = \mathbf{F}_m\, \ddot{\mathbf{R}}_{m-1,m} \left\{\mathbf{S}^>_m + \mathbf{S}^{uu}_{mm}\right\} \mathbf{F}_m \tag{42}$$

where $\mathbf{F}_m = diag\left\{e^{iq_h d_m}, e^{iq_v d_m}\right\}$. Notice that this is an operator relation where all elements are operators. Taking the tensor product of (42) with its complex conjugate we have

$$\mathbf{S}^{du}_{mm} \otimes \mathbf{S}^{du*}_{mm} = (\mathbf{F}_m \otimes \mathbf{F}^*_m)\left(\dddot{\mathbf{R}}_{m-1,m} \otimes \dddot{\mathbf{R}}^*_{m-1,m}\right)\left(\left\{\mathbf{S}^>_m + \mathbf{S}^{uu}_{mm}\right\} \otimes \left\{\mathbf{S}^>_m + \mathbf{S}^{uu}_{mm}\right\}^*\right)(\mathbf{F}_m \otimes \mathbf{F}^*_m). \tag{43}$$

Next we average (43) w.r.t. surface fluctuations and get

$$\left\langle \mathbf{S}^{du}_{mm} \otimes \mathbf{S}^{du*}_{mm} \right\rangle \simeq (\mathbf{F}_m \otimes \mathbf{F}^*_m)\left\langle \dddot{\mathbf{R}}_{m-1,m} \otimes \dddot{\mathbf{R}}^*_{m-1,m}\right\rangle \left\langle \left\{\mathbf{S}^>_m + \mathbf{S}^{uu}_{mm}\right\} \otimes \left\{\mathbf{S}^>_m + \mathbf{S}^{uu}_{mm}\right\}^* \right\rangle (\mathbf{F}_m \otimes \mathbf{F}^*_m) \tag{44a}$$

where we have approximated that the two tensor products in the middle are weakly correlated. A further approximation that we impose is given as follows

$$\left\langle \left\{\mathbf{S}^>_m + \mathbf{S}^{uu}_{mm}\right\} \otimes \left\{\mathbf{S}^>_m + \mathbf{S}^{uu}_{mm}\right\}^* \right\rangle \simeq \mathbf{S}^>_m \otimes \mathbf{S}^{>*}_m + \left\langle \mathbf{S}^{uu}_{mm} \otimes \mathbf{S}^{uu*}_{mm}\right\rangle. \tag{44b}$$

These are the kinds of approximations required to arrive at our boundary conditions.

5. TRANSITION TO RADIATIVE TRANSFER

Now we have to transition from this transport equation (44) to the phenomenological radiative transfer equation discussed earlier. To accomplish this we have to link the key quantities of waves and radiative transfer, viz., coherence function and specific intensity. The relation between them is obtained by computing the energy density using the two concepts. Thus we have

$$\frac{1}{2}\epsilon\left\langle E_\mu(\mathbf{r})E^*_\nu(\mathbf{r})\right\rangle = \sqrt{\mu\epsilon}\int d\Omega_s I_{\mu\nu}(\mathbf{r},\hat{s}). \tag{45}$$

Wigner transform provides us following relation:

$$\left\langle E_\mu(\mathbf{r})E^*_\nu(\mathbf{r})\right\rangle = \frac{1}{(2\pi)^2}\int d\mathbf{k}_\perp \mathcal{E}_{\mu\nu}(z,\mathbf{k}_\perp). \tag{46}$$

From (45) and (46) we relate I to $\mathcal{E}$ as

$$I_{\mu\nu}(z,\hat{s}) = \frac{1}{2\eta}\frac{{k'}^2}{(2\pi)^2}\cos\theta\mathcal{E}_{\mu\nu}(z,\mathbf{k}_\perp). \tag{47}$$

There is still one difference that needs to be ironed out before we transition to the RT equations. Notice that in our wave approach we obtained transport equations for $\tilde{\mathcal{E}}$, which is the fluctuating part of the coherence function. On the other hand the phenomenological RT equations are traditionally written for total intensities. Therefore we have to express our transport equations in terms of $\mathcal{E}$. Notice that $\mathcal{E} = \bar{\mathcal{E}} + \tilde{\mathcal{E}}$, where $\bar{\mathcal{E}}$, the average part of $\mathcal{E}$ satisfies:

$$\left\{\frac{d}{dz} - ia(q_\mu - q^*_\nu)\right\}\bar{\mathcal{E}}^a_m(z,\mathbf{k}_\perp) = 0, \tag{48}$$

Using (48) in (39) we obtain

$$\left\{\frac{d}{dz} + i\left[q_\mu(\mathbf{k}_\perp) - q^*_\nu(\mathbf{k}_\perp)\right]\right\}\mathcal{E}^u_{\mu\nu}(z,\mathbf{k}_\perp) = \frac{|k_m|^4}{(2\pi)^2}\int d\boldsymbol{\alpha}_\perp \mathbf{S}^>_\mu \otimes \mathbf{S}^{>*}_\nu$$
$$\Phi_m\left\{\mathbf{k}_\perp - \boldsymbol{\alpha}_\perp; \frac{1}{2}\left[q_\mu(\mathbf{k}_\perp) - q^*_\nu(\mathbf{k}_\perp)\right] - \frac{1}{2}a\left[q_{\mu'}(\boldsymbol{\alpha}_\perp) - q^*_{\nu'}(\boldsymbol{\alpha}_\perp)\right]\right\}\mathcal{E}^a_{\mu'\nu'}(z,\boldsymbol{\alpha}_\perp), \tag{49a}$$

$$\left\{-\frac{d}{dz} + i\left[q_\mu(\mathbf{k}_\perp) - q^*_\nu(\mathbf{k}_\perp)\right]\right\}\mathcal{E}^d_{\mu\nu}(z,\mathbf{k}_\perp) = \frac{|k_m|^4}{(2\pi)^2}\int d\boldsymbol{\alpha}_\perp \mathbf{S}^<_\mu \otimes \mathbf{S}^{<*}_\nu$$
$$\Phi_m\left\{\mathbf{k}_\perp - \boldsymbol{\alpha}_\perp; -\frac{1}{2}\left[q_\mu(\mathbf{k}_\perp) - q^*_\nu(\mathbf{k}_\perp)\right] - \frac{1}{2}a\left[q_{\mu'}(\boldsymbol{\alpha}_\perp) - q^*_{\nu'}(\boldsymbol{\alpha}_\perp)\right]\right\}\mathcal{E}^a_{\mu'\nu'}(z,\boldsymbol{\alpha}_\perp), \tag{49b}$$

Notice that this equation is expressed entirely in total intensity. Now we can transition to the phenomenological RT equations. Using the relation between $\mathcal{E}$ and I we change the integration variable to solid angle and arrive at the following equation,

$$\left\{\cos\theta\frac{d}{dz}+\eta_{ij}\right\}I_j^u(z,\hat{s})=\int d\Omega' P_{ij}^{ua}(\Omega,\Omega')I_j^a(z,\hat{s}'), \tag{50a}$$

$$\left\{-\cos\theta\frac{d}{dz}+\eta_{ij}\right\}I_j^d(z,\hat{s})=\int d\Omega' P_{ij}^{da}(\Omega,\Omega')I_j^a(z,\hat{s}'), \tag{50b}$$

where $\overline{\overline{\eta}}$ is the extinction matrix and $\overline{\overline{\mathbf{P}}}$ is the phase matrix. Implicit summation over superscripts a and subscript ν is assumed in (50). To facilitate comparison with the results of Ulaby et al.,[6] and Lam and Ishimaru[7] we have used a modified version of Stokes vector.[17] Instead of the standard form $\{I,Q,U,V\}$ we use $\{(I+Q)/2,(I-Q)/2,U,V\}$. The subscript of I denote the element number of our Stokes vector. Although the structure of this equation is identical to that of RT (equation (2)) the elements of the phase matrix and the extinction matrices are not the same. The primary reason is because of the differences in the real part of the mean propagation constants of horizontally and vertically polarized waves. On assuming that $q_h'=q_v'=k_{mz}'$ we obtain the following expressions for the extinction and phase matrices:

$$\overline{\overline{\eta}}=-\cos\theta\ \mathrm{diag}\left\{2q_v'',2q_h'',q_v''+q_h'',q_v''+q_h''\right\} \tag{51a}$$

$$P_{\mu\nu}^{ab}=\frac{1}{(2\pi)^2}\frac{1}{4}|k_m|^4\Phi_m\left\{\mathbf{k}_\perp-\mathbf{k}'_\perp;k_m'[a\cos\theta-b\cos\theta']\right\}\mathcal{P}_{\mu\nu}^{ab} \tag{51b}$$

For $\mu=v,h$,

$$\begin{aligned}\mathcal{P}_{\mu v}^{ab}&=\left(\boldsymbol{\mu}^a\cdot\mathbf{v}'^b\right)^2 & \mathcal{P}_{\mu h}^{ab}&=\left(\boldsymbol{\mu}^a\cdot\mathbf{h}'^b\right)^2\\ \mathcal{P}_{\mu U}^{ab}&=\left(\boldsymbol{\mu}^a\cdot\mathbf{v}'^b\right)\left(\boldsymbol{\mu}^a\cdot\mathbf{h}'^b\right) & \mathcal{P}_{\mu V}^{ab}&=0\end{aligned} \tag{52}$$

Similarly,

$$\begin{aligned}\mathcal{P}_{Uv}^{ab}&=2\left(\mathbf{v}^a\cdot\mathbf{v}'^b\right)\left(\mathbf{h}^a\cdot\mathbf{v}'^b\right) \qquad \mathcal{P}_{Uh}^{ab}=2\left(\mathbf{v}^a\cdot\mathbf{h}'^b\right)\left(\mathbf{h}^a\cdot\mathbf{h}'^b\right)\\ \mathcal{P}_{UU}^{ab}&=\left(\mathbf{v}^a\cdot\mathbf{v}'^b\right)\left(\mathbf{h}^a\cdot\mathbf{h}'^b\right)+\left(\mathbf{v}^a\cdot\mathbf{h}'^b\right)\left(\mathbf{h}^a\cdot\mathbf{v}'^b\right) \qquad \mathcal{P}_{UV}^{ab}=0\end{aligned} \tag{53}$$

$$\begin{aligned}\mathcal{P}_{Vv}^{ab}&=\mathcal{P}_{Vh}^{ab}=\mathcal{P}_{VU}^{ab}=0\\ \mathcal{P}_{VV}^{ab}&=\left(\mathbf{v}^a\cdot\mathbf{v}'^b\right)\left(\mathbf{h}^a\cdot\mathbf{h}'^b\right)-\left(\mathbf{v}^a\cdot\mathbf{h}'^b\right)\left(\mathbf{h}^a\cdot\mathbf{v}'^b\right)\end{aligned} \tag{54}$$

Noting the implied summation over a in (50) we see that they are identical to the RT equations given in Section 2. Now we have explicit expressions for the extinction matrix and phase matrix in terms of the statistical parameters of the problem thanks to our wave approach. We next turn our attention to the boundary conditions. In our wave approach we obtained BCs in terms of 'stack' reflection matrix $\dddot{\mathbf{R}}$, whereas in the RT approach the BCs are given in terms of the local interface reflection matrices. We can readily reconcile with this apparent difference. Note that the BC in the wave approach forms a closed system whereas in the RT approach it is open (linked to adjacent layer intensities). Let us take a look at the BC at the $(m-1)$-th interface. $\dddot{\mathbf{R}}_{m-2,m}$ can be expressed in terms of $\mathbf{R}_{m-2,m-1}$ as follows,

$$\dddot{\mathbf{R}}_{m-1,m}=\mathbf{R}_{m-1,m}+\mathbf{T}_{m,m-1}\left\{I-\dddot{\mathbf{R}}_{m-2,m-1}\,F_{m-1}\mathbf{R}_{m,m-1}\right\}^{-1}\dddot{\mathbf{R}}_{m-2,m-1}\,F_{m-1}\mathbf{T}_{m-1,m}. \tag{55}$$

This is the relation between the stack reflection coefficients of adjacent interfaces. The $\mathbf{R}$ and $\mathbf{T}$ are local (single interface) reflection and transmission matrices at the $(m-1)$-th interface. On operating $\mathbf{E}_m^u$ with (54) we get

$$\mathbf{E}_m^d=\mathbf{R}_{m-1,m}\mathbf{E}_m^u+\mathbf{T}_{m,m-1}\mathbf{E}_{m-1}^d. \tag{56}$$

Notice that this boundary condition now involves only local interface Fresnel coefficients.

Similarly we write $\dddot{\mathbf{R}}_{m+1,m}$ in terms of $\dddot{\mathbf{R}}_{m+2,m+1}$ and hence obtain the BC at the m-th interface

$$\mathbf{E}_m^u = \mathbf{R}_{m+1,m}\mathbf{E}_m^d + \mathbf{T}_{m,m+1}\mathbf{E}_{m+1}^u. \tag{57}$$

Take the tensor product of (56) with its complex conjugate and average w.r.t. surface fluctuations. Employing the Wigner transform operator, we obtain boundary condition at the $(m-1)$-th interface similar to that of the RT system. However, the reflection and transmission matrices used in the RT system correspond to unperturbed medium as opposed to the average medium as in the case of the wave approach.

6. DISCUSSION

Now that we have made the transition from statistical wave theory to radiative transfer theory it is instructive to itemize the assumptions implicitly involved in the RT approach.
1. Weak fluctuations
2. Quasi-stationary field approximation.
3. Statistical homogeneity of fluctuations.

These are the three well-known conditions necessary for the unbounded random medium problem. However, if the medium is bounded we need to impose additional conditions. We found that the extinction coefficients calculated in the wave approach and the RT approach are different and only after applying further approximations can they be made to agree with each other. The following two additional conditions are required for bounded random media:
4. While calculating effective propagation constants we ignore terms involving interface reflection coefficients. In other words we ignore all boundary effects.
5. While carrying out Wigner transforms of the Green's functions we need to neglect the interactions between components.
6. Use unperturbed medium parameters while imposing boundary conditions.

The above two conditions are required when the boundaries are planar. However, if they are randomly rough we need to further impose two more conditions:
7. Weak surface correlation approximation.
8. All fluctuations of the problem are statistically independent.

To summarize, we have enquired into the assumptions involved in adopting the radiative transfer approach to scattering from layered random media with rough interfaces. To facilitate this enquiry we adopted a wave approach to this problem and derived the governing equations for the first and second moments of the wave fields. We employed Wigner transforms and transitioned to the system corresponding to that of radiative transfer approach. In this process we found that there are more conditions implicitly involved in the RT approach to this problem than it widely believed to be necessary. With the recent development of fast and efficient algorithms for scattering computations and the enormous increase in computer resources it now feasible to take an entirely numerical approach to this problem without imposing any approximations. However, to keep the size of the problem to be manageable only special cases have been studied thus far.[22–24] Hence it is very much of relevance and interest to use the RT approach to these problems. However, one should keep in mind the assumptions involved in such an approach.

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